Tuning Distributed Control Algorithms for Optimal Functioning

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Abstract. In this paper, we present a model which characterizes distributed computing algorithms. The goals of this model are to offer an abstract representation of asynchronous and heterogeneous distributed systems, to present a mechanism for specifying externally observable behaviours of distributed processes and to provide rules for combining these processes into networks with desired properties (good functioning, fairness . . .). Once these good properties are found, the determination of the optimal rules are studied.

Subsequently, the model is applied to three classical distributed computing problems: namely the dining philosophers problem, the mutual exclusion problem and the deadlock problem, (generalizing results of our previous publications [1], [2]). The property of fairness has a special position that we discuss.

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1. Introduction

This paper presents a new model which describes the phenomena of distributed computing through a behavioural study of distributed algorithms. This model is based on the interconnection of N Markov chains, each representing a distributed process.

Our model differs from the usual ones (see [4], [5], [14]) since it handles a formal specification of distributed systems through local considerations. Good functioning properties for the execution of algorithms are found from the processes' behaviour. Thus good rules for designing algorithms with desired properties (liveness, safety, fairness) are found.

Optimal functioning is then studied through the optimization of a decision function under some constraints imposed by the network structure.

The model is applied to three classical problems: namely, the dining philosophers problem, the mutual exclusion problem and the deadlock problem in which fairness reveals itself to be an important issue.

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2. Formal Model for Distributed Algorithms

A distributed system is a software and hardware structure distributed in a network of processes which computes information by message exchange. This network consists of sites (or processes) and a communication system. Each site corresponds, at the hardware level, to a processor with its own local memory and, at the software level, to a sequential process with communication primitives ([7], [8], [10], [12]).

The emergence of real distributed machines has added a new urgency to the development of adaptable algorithms and to their control. In order to solve the problems encountered in the development of distributed operating systems, a formal model must be defined. In particular, distributed mechanisms must be well understood, new problems related to the use of parallelism have to be highlighted, and adapted solutions have to be found ([12], 13]).

Three main issues are usually considered over this framework on distributed control problems: the design of control algorithms, their correctness proof and the evaluation of their performance. Our model helps to investigate the last two issues as it will be described in the following. Its main characteristics are that it is based on an observational approach, it deals only with local considerations and uses a known mathematical tool, the Markov chains.

This gives a new approach to deal with distributed systems.

3. Decision Function and Optimality Criteria

Consider N finite homogeneous Markov chains with state spaces ${}^k\mathcal{X} = \{1, \ldots, \nu_k\}$, $(k = 1, \ldots, N)$ and corresponding transition matrices ${}^kM$, $(k = 1, \ldots, N)$.

The notation ${}^kM$ expresses the fact that each transition matrix depends on a multi-dimensional parameter ρ_k characterizing it, for example $\rho_k = ({}^kp_{11}, \ldots, {}^kp_{ij}, \ldots, {}^kp_{\nu_k\nu_k})$. Thus, these ${}^kM$ are matrices M_{ρ_k}.

DEFINITIONS

(i) The distributed system is made up of a network of processes logically represented by the interconnection of N Markov chains. Then there exists a set of relations between the parameters $\rho_1, \ldots, \rho_N$ which defines and characterizes the network,

$$\mathcal{R}_j(\rho_1, \ldots, \rho_N) = 0 \quad j = 1, \ldots, N. \tag{1}$$

These relations are *constraints* (which can be linear). We write $\rho = (\rho_1, \ldots, \rho_N) \in \mathcal{R}$ iff the parameters $\rho_1, \ldots, \rho_N$ satisfy (1).

(ii) For every $k \in \{1, \ldots, N\}$, suppose that there exists only one acyclic ergodic class ${}^k\mathcal{E}$ and at most one transient class ${}^k\mathcal{P}$. This is the *condition* (C).

For the chain number k, if $i \in {}^k\mathscr{E}$, let ${}^kT_i$ be the mean recurrence time of state i (that is to say ${}^kT_i = \Sigma_{n=1}^{\infty} nf_{ii}^{(n)}$, where $f_{ii}^{(n)}$ is the probability that starting from state i, one comes back to i for the first time in n steps). Recall that ${}^kT_i = 1/\lim_{n\to\infty} {}^km_{ji}^{(n)}$, this latter expression is independent of j, ${}^km_{ji}^{(n)}$ being the coefficient (j, i) of the matrix $({}^kM)^n$.

If j and $j' \in {}^k\mathscr{P}$, let us denote by ${}^kS_{jj'}$ the total mean sojourn time in j' starting from j. These ${}^kS_{jj'}$ are given by the matrix $(I - {}^kW)^{-1}$ where ${}^kW$ is the submatrix restriction of ${}^kM$ to the transient states (states of ${}^k\mathscr{P}$). A *decision function* is a real function f of variables ${}^kT_i$ and ${}^kS_{jj'}$, $i \in {}^k\mathscr{E}$, j and $j' \in {}^k\mathscr{P}$, and with $k \in \{1, \ldots, N\}$. Since these ${}^kT_i$ and ${}^kS_{jj'}$ are expressed in term of matrices $({}^kM)^n$, i.e., in term of ρ_k, f is a function of the variables ρ_k. The definition of a decision function f implies that the network verifies condition (C). For each problem, following its context, a decision function will be defined and its role will be to control the functioning of the system and to find so the optimum policies.

(iii) An optimality criterion is based on the optimization (maximization or minimization) of a decision function f, under some constraints.

4. Our Model

Referring to many authors (see for example [4], [5], [7] [12], [14]), we specify our definition of a distributed system.

Our model $(\mathscr{S}, \mathscr{S}^0, \mathscr{T}, \mathscr{G})$ is based upon the interconnection of N Markov chains:

- $\mathscr{S}$, the set of *system-states*, is here the set $\Pi_{k=1}^{N}({}^k\mathscr{X})$.
- $\mathscr{S}^0$, the set of *initial system-states* is a subset of $\mathscr{S}$.
- $\mathscr{T}$ is the set of *functioning rules*. Each functioning rule, denoted here by M_ρ, is a N-tuple of transition matrices $(M_{\rho_1}, \ldots, M_{\rho_N})$, where $\rho \in \mathscr{R}$. We are only interested in functioning rules with which we can associate a decision function f (that is to say only with rules associated with a Markovian network which satisfies condition (C)).

  A functioning rule is said to be *optimal* if and only if its ρ maximizes (resp. minimizes) the decision function f when the imposed optimality criterion is the maximization (resp. minimization) of f.

  A functioning rule M_ρ is said to be *bad* if and only if its ρ maximizes (resp. minimizes) the decision function f when the imposed optimality criterion is the minimization (resp. maximization) of f. Every functioning rule which is not bad is said to be *advisable*. Optimal and advisable functioning rules are *good* functioning rules.
- For each problem, we want to find one or several optimal functioning rules, or if this is not possible to find advisable ones. We let $\mathscr{G} = \{M_\rho,\ \rho \in \mathscr{R}_0,\ \mathscr{R}_0 \subset \mathscr{R}\} \subset \mathscr{T}$, define the set of good functioning rules of the problem.

  Now, we are going to study 3 applications of our model.

5. The Dining Philosophers' Problem

5.1. THE PROBLEM

In distributed computing, the problem of resource allocation and of solving conflicts between processes is well illustrated by the dining philosophers problem. Traditionally, the philosophers are arranged in a circle around a spaghettis plate, with a fork between each pair of philosophers. In order to eat, each philosopher requires his two adjacent forks ([7]). Algorithmic solutions can be found in [7].

The problem considers a network of N finite homogeneous Markov chains with the same state space. $\forall k \in \{1, \ldots, N\}$, ${}^k\mathcal{X} = \mathcal{X} = \{1, \ldots, 4\}$, where state 1 = "waiting", state 2 = "with one fork", state 3 = "with two forks", state 4 = "thinking", and with the following transition matrix:

$$M_{\rho_k} = \begin{pmatrix} 1-\alpha_k & \alpha_k & 0 & 0 \\ 0 & 1-\beta_k & \beta_k & 0 \\ 0 & 0 & 1-\gamma_k & \gamma_k \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $\rho_k = (\alpha_k, \beta_k, \gamma_k) \in]0, 1[^3$, for $k \in \{1, \ldots, N\}$.

The Markov chains verify condition (C): more precisely, each of these chains has only one acyclic ergodic class and no transient class. The connection in the network is expressed here by the relations:

$$L_k(\alpha_k, \beta_{k+1}) \equiv \alpha_k + \beta_{k+1} - 1 = 0 \quad (k = 1, \ldots, N-1)$$

$$L_N(\alpha_N, \beta_1) \equiv \alpha_N + \beta_1 - 1 = 0 .$$

The set of $\rho_k = (\alpha_k, \beta_k, \gamma_k)$ verifying these constraints is denoted by $\mathcal{R}$.

5.2. DEFINITIONS

The set $\mathcal{S}$ of system-states is here $\mathcal{X}^N$.
The set $\mathcal{S}^0$ of initial system-states is here the set $\mathcal{X}^N$ itself.
The set $\mathcal{T}$ of functioning rules is here $(M_\rho)_{\rho \in \mathcal{R}}$.
The decision function is here:

$$f = \sum_{k=1}^{N} q_k \frac{\alpha_k \beta_k + \beta_k \gamma_k + \gamma_k \alpha_k + \alpha_k \beta_k \gamma_k}{\alpha_k \beta_k},$$

where the constants q_k are such that $\forall k \in \{1, \ldots, N\}$, $q_k \in]0, 1[$ and $\Sigma_{k=1}^N q_k = 1$. The reason to the choice of this function is that: on the one hand, $\forall k \in \{1, \ldots, N\}$, each state is recurrent so that the ${}^kT_i$, $(i = 1, 2, 3, 4)$, have a meaning; on the other hand, the best functioning rule is the one that minimizes the mean recurrence time to the state with two forks (that is to say, the rule that performs a return to state 3 the most often possible). Thus, with the help of the computation of $\lim_{n\to\infty}({}^kM)^n$, we obtain the expression for the $\frac{1}{{}^kT_i}$; in particular,

$$\frac{1}{{}^kT_3} = \frac{\alpha_k \beta_k}{\alpha_k \beta_k + \beta_k \gamma_k + \gamma_k \alpha_k + \alpha_k \beta_k \gamma_k} .$$

The function f expresses the sum of the recurrence time ${}^kT_3$ weighted by the coefficients q_k. Its minimization determines the desired functioning rule.

5.3. RESULTS

THEOREM 5.1. $\forall(\gamma_1, \ldots, \gamma_N) \in (]0, 1[)^N$, *the unique optimal functioning rule is* $M_\rho = (M_{\rho_1}, \ldots, M_{\rho_N})$ *where*

$$\rho_k = \left(\frac{1}{1 + \sqrt{\dfrac{q_{k+1}\gamma_{k+1}}{q_k \gamma_k}}}, \frac{\sqrt{\dfrac{q_{k+1}\gamma_{k+1}}{q_k\gamma_k}}}{1 + \sqrt{\dfrac{q_{k+1}\gamma_{k+1}}{q_k\gamma_k}}}, \gamma_k \right) \qquad k \in \{1, \ldots, N-1\}$$

and

$$\rho_N = \left(\frac{1}{1 + \sqrt{\dfrac{q_1\gamma_1}{q_N\gamma_N}}}, \frac{\sqrt{\dfrac{q_1\gamma_1}{q_N\gamma_N}}}{1 + \sqrt{\dfrac{q_1\gamma_1}{q_N\gamma_N}}}, \gamma_N \right).$$

If the $\gamma_1, \ldots, \gamma_N$ *are bounded from below, that is to say* $(\gamma_1, \ldots, \gamma_N) \in \Pi_{k=1}^N [c_k, 1[$, *then the set* $\mathcal{G}$ *of good functioning rules is the singleton* $\{M_\rho\}$, *where*

$$\rho_k = \left(\frac{1}{1 + \sqrt{\dfrac{q_{k+1}c_{k+1}}{q_k c_k}}}, \frac{\sqrt{\dfrac{q_{k+1}c_{k+1}}{q_k c_k}}}{1 + \sqrt{\dfrac{q_{k+1}c_{k+1}}{q_k c_k}}}, c_k \right) \qquad k \in \{1, \ldots, N-1\}$$

and

$$\rho_N = \left(\frac{1}{1 + \sqrt{\dfrac{q_1c_1}{q_Nc_N}}}, \frac{\sqrt{\dfrac{q_1c_1}{q_Nc_N}}}{1 + \sqrt{\dfrac{q_1c_1}{q_Nc_N}}}, c_N \right).$$

Proof. By considering the constraints L_j and introducing the Lagrange multipliers λ_j, $(j \in \{1, \ldots, N\})$, we have to solve the following system of equations:

$$\frac{\partial f}{\partial \alpha_k} + \frac{\partial}{\partial \alpha_k} \sum_{k=1}^N \lambda_j L_j \equiv \frac{-q_k\gamma_k}{\alpha_k^2} + \lambda_k = 0 , \quad k \in \{1, \ldots, N\} ,$$

$$\frac{\partial f}{\partial \beta_k} + \frac{\partial}{\partial \beta_k} \sum_{k=1}^N \lambda_j L_j \equiv \frac{-q_k\gamma_k}{\beta_k^2} + \lambda_{k-1} = 0 , \quad k \in \{2, \ldots, N\} ,$$

$$\frac{\partial f}{\partial \beta_1} + \frac{\partial}{\partial \beta_1} \sum_{k=1}^N \lambda_j L_j \equiv \frac{-q_1\gamma_1}{\beta_1^2} + \lambda_N = 0 ,$$

which implies

$$\frac{q_k \gamma_k}{\alpha_k^2} = \frac{q_{k+1}\gamma_{k+1}}{\beta_{k+1}^2},$$

and thus

$$\beta_{k+1} = \sqrt{\frac{q_{k+1}\gamma_{k+1}}{q_k \gamma_k}}\,\alpha_k .$$

Since $\alpha_k + \beta_k = 1$, we have

$$\alpha_k = \frac{1}{1 + \sqrt{\frac{q_{k+1}\gamma_{k+1}}{q_k\gamma_k}}},$$

and

$$\beta_k = \frac{\sqrt{\frac{q_{k+1}\gamma_{k+1}}{q_k\gamma_k}}}{1 + \sqrt{\frac{q_{k+1}\gamma_{k+1}}{q_k\gamma_k}}}.$$

Moreover, the function

$$(\alpha_1, \beta_1, \ldots, \alpha_N, \beta_N) \to \sum_{k=1}^{N} q_k\left[(1+\gamma_k) + \frac{\gamma_k}{\alpha_k} + \frac{\gamma_k}{\beta_k}\right] + \sum_{k=1}^{N-1} \lambda_k L_k(\alpha_k, \beta_k) + \lambda_N L_N(\alpha_N, \beta_1)$$

is convex as a sum of convex functions. Thus, the optimum is a minimum.

When $(\gamma_1, \ldots, \gamma_N)$ varies, and if the parameters are bounded, the form of f shows that the minimum is reached for $(\gamma_1, \ldots, \gamma_N) = (c_1, \ldots, c_N)$.

Remark that, in the particular case where $\forall k \in \{1, \ldots, N\}$ $q_k = \frac{1}{N}$, we find the result reported in [1] with

$$\rho_k = \left(\frac{1}{1+\sqrt{\frac{c_{k+1}}{c_k}}}, \frac{\sqrt{\frac{c_{k+1}}{c_k}}}{1+\sqrt{\frac{c_{k+1}}{c_k}}}, c_k\right).$$

■

6. The Mutual Exclusion Problem

6.1. THE PROBLEM

When many processes require an access to shared resources, mutual exclusion must be ensured. Such a protocol consists in a policy which allows at most one process to work with the resource. This is the typical contention problem and an overview of solutions can be found in [7], [11].

The problem considers a network of N homogeneous and finite Markov chains with the same state space $\forall k \in \{1, \ldots, N\}$, ${}^k\mathscr{X} = \mathscr{X} = \{1, 2, 3, 4\}$, where state 1 = "request state", state 2 = "refusal", state 3 = "acceptation", state 4 = "execution", and with the following transition matrix:

$$M_{\rho_k} = \begin{pmatrix} \alpha_k & \beta_k & 1-\alpha_k-\beta_k & 0 \\ 1-\gamma_k & \gamma_k & 0 & 0 \\ 0 & 0 & \delta_k & 1-\delta_k \\ 1-\theta_k & 0 & 0 & \theta_k \end{pmatrix},$$

where $\rho_k = (\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k) \in]0, 1[^5$ and $1 - \alpha_k - \beta_k > 0$. The Markov chains verify condition (C): each of the chains has one and only one acyclic ergodic class (and has no transient class). The interconnection into network is expressed by the relations:

$$L_1 \equiv \sum_{k=1}^{N} \alpha_k - 1 = 0$$

(there is almost surely a "request"), and

$$L_2 \equiv \sum_{k=1}^{N} (1 - \alpha_k - \beta_k) - 1 = 0$$

(there is almost surely an "acceptation"). The latter relation can also be written

$$L_2 \equiv N - 2 - \sum_{k=1}^{N} \beta_k = 0\,,$$

which shows that $N > 3$.

The set of $\rho_k = (\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k)$ verifying these constraints is denoted by $\mathscr{R}$.

6.2. DEFINITIONS

The set $\mathscr{S}$ of system states is $\mathscr{X}^N$.
The set $\mathscr{S}^0$ of initial system states is the set $\mathscr{X}^N$ itself.
The set $\mathscr{T}$ of functioning rules is $(M_\rho)_{\rho \in \mathscr{R}}$.
The decision function is:

$$f = \sum_{k=1}^{N} q_k \frac{{}^kT_2}{{}^kT_3} = \sum_{k=1}^{N} q_k \frac{(1-\gamma_k)}{(1-\delta_k)} \frac{(1-\alpha_k-\beta_k)}{\beta_k}\,,$$

where the constants q_k are such that $\forall k \in \{1, \ldots, N\}$, $q_k \in]0, 1[$ and $\Sigma_{k=1}^{N} q_k = 1$.

As in the previous problem, from the computation of $\lim_{n\to\infty} ({}^kM)^n$, we obtain the expression for $\frac{1}{{}^kT_i}$; in particular:

$${}^kT_2 = 1 + \frac{1-\gamma_k}{\beta_k} + \frac{1-\alpha_k-\beta_k}{\beta_k(1-\delta_k)} + \frac{(1-\alpha_k-\beta_k)(1-\gamma_k)}{\beta_k(1-\theta_k)}\,,$$

and

$$^{k}T_3 = 1 + \frac{1-\delta_k}{1-\alpha_k-\beta_k} + \frac{\beta_k(1-\delta_k)}{(1-\alpha_k-\beta_k)(1-\gamma_k)} + \frac{1-\delta_k}{1-\theta_k}\,.$$

In the present case, the best functioning rule consists in reducing as much as possible the mean recurrence time of the state "refusal" (that is to say, coming back as *often* as possible to the state "*refusal*") and at the same time in maximizing the mean recurrence time of the state "acceptation" (that is to say, coming back as late as possible to the state "acceptation"). Minimizing the sum of the ratios of the mean recurrence time of the state "refusal" to the mean recurrence time of the state "acceptation" corresponds to the desired functioning rule.

6.3. RESULTS

THEOREM 6.1. (a) $\forall(\gamma_1, \delta_1, \theta_1, \ldots, \gamma_N, \delta_N, \theta_N) \in (]0, 1[)^{3N}$, *the unique optimal functioning rule is* $M_\rho = (M_{\rho_1}, \ldots, M_{\rho_N})$, *where*

$$\rho_k = \left(1 - \frac{(N-1)q_k\frac{1-\gamma_k}{1-\delta_k}}{\sum_{j=1}^{N} q_j\frac{1-\gamma_j}{1-\delta_j}},\ \frac{(N-2)q_k\frac{1-\gamma_k}{1-\delta_k}}{\sum_{j=1}^{N} q_j\frac{1-\gamma_j}{1-\delta_j}},\ \gamma_k, \delta_k, \theta_k\right).$$

(b) *If the* $\gamma_1, \ldots, \gamma_N$ *are bounded from above and the* $\delta_1, \ldots, \delta_N$ *are bounded from below, that is to say if* $\gamma_k \in]0, c_k]$ *and* $\delta_k \in [d_k, 1[$, $k \in \{1, \ldots, N\}$. *Then the set* $\mathcal{G}$ *of good functioning rules is the set of optimal functioning rules* M_ρ *described in* (a) *with* $(\gamma_k, \delta_k) = (c_k, d_k)$, $k \in \{1, \ldots, N\}$, *when* $(\theta_1, \ldots, \theta_N)$ *varies in* $(]0, 1[)^N$.

(c) *In the particular case where* $\forall k \in \{1, \ldots N\}$, $(c_k, d_k) = (c, d)$, $\theta_k = \frac{1}{N}$, *and* $q_k = \frac{1}{N}$, *then we find the fairness solution which is*

$$\rho_k = \left(\frac{1}{N}, \frac{N-2}{N}, c, d, \frac{1}{N}\right).$$

Proof. By considering the constraints L_1 and L_2, and introducing the Lagrange multipliers λ_1 and λ_2, we have to solve the following system of equations:

$$\frac{\partial f}{\partial \alpha_k} + \lambda_1 \frac{\partial L_1}{\partial \alpha_k} + \lambda_2 \frac{\partial L_2}{\partial \alpha_k} \equiv \frac{q_k(1-\gamma_k)}{\beta_k(1-\delta_k)} - \lambda_1 = 0\,, \tag{1}$$

$$\frac{\partial f}{\partial \beta_k} + \lambda_1 \frac{\partial L_1}{\partial \beta_k} + \lambda_2 \frac{\partial L_2}{\partial \beta_k} \equiv -\frac{q_k(1-\gamma_k)(1-\alpha_k)}{\beta_k^2(1-\delta_k)} - \lambda_2 = 0\,. \tag{2}$$

The solution of equation (1) gives

$$\lambda_1 = \frac{q_k\frac{1-\gamma_k}{1-\delta_k}}{\beta_k} = \frac{\sum_{j=1}^{N} q_j\frac{1-\gamma_j}{1-\delta_j}}{N-2}\,,$$

viz.

$$\beta_k = \frac{(N-2)q_k \frac{1-\gamma_k}{1-\delta_k}}{\Sigma_{j=1}^N q_j \frac{1-\gamma_j}{1-\delta_j}} .$$

Considering this value of β_k, (2) gives:

$$-\lambda_2 = \frac{(1-\alpha_k)\, \Sigma_{j=1}^N q_j \frac{1-\gamma_j}{1-\delta_j}}{(N-2)^2 q_k \frac{1-\gamma_k}{1-\delta_k}} = \frac{(N-1)}{(N-2)^2} ,$$

which implies

$$\alpha_k = 1 - \frac{(N-1)q_k \frac{1-\gamma_k}{1-\delta_k}}{\Sigma_{j=1}^N q_j \frac{1-\gamma_j}{1-\delta_j}} .$$

Let us now show that this solution corresponds to a minimum for f. To this end, let us reduce the initial problem to an equivalent one by setting $w_k = q_k \frac{1-\gamma_k}{1-\delta_k}$ and $A_k^2 = 1-\alpha_k$. The equivalent problem is to minimize

$$F(A_1, \beta_1, \ldots, A_N, \beta_N) = \sum_{k=1}^N w_k \frac{A_k^2}{\beta_k}$$

under the constraints

$$N - 2 - \sum_{k=1}^N \beta_k = 0 ,$$

$$\sum_{k=1}^N A_k^2 = N - 1 ,$$

$$A_k^2 < 1 , \quad \beta_k > 0 , \quad A_k^2 - \beta_k > 0 .$$

It is straightforward that the solutions

$$\hat{\alpha}_k = 1 - \frac{(N-1)w_k}{\Sigma_{j=1}^N w_j} , \hat{\beta}_k = \frac{(N-2)w_k}{\Sigma_{j=1}^N w_j}$$

of the initial problem correspond to the solutions

$$\hat{A}_k = \sqrt{\frac{(N-1)w_k}{\Sigma_{j=1}^N w_j}} , \hat{\beta}_k = \frac{(N-2)w_k}{\Sigma_{j=1}^N w_j}$$

of the equivalent problem. Note that the function

$$(A_k, \beta_k) \to \frac{A_k^2}{\beta_k}$$

is convex, since its Hessian

$$\begin{pmatrix} \dfrac{2}{\beta_k} & \dfrac{-2A_k}{\beta_k^2} \\ \dfrac{-2A_k}{\beta_k^2} & \dfrac{2A_k}{\beta_k^2} \end{pmatrix}$$

corresponds to a positive semi-definite quadratic form. F is thus convex as a linear combination of convex functions with positive coefficients. Consequently, the solution $(\hat{A}_1, \hat{\beta}_1, \ldots, \hat{A}_N, \hat{\beta}_N)$ corresponds to a minimum for F, and it is the same for the solution

$$\left(\frac{1-(N-1)w_k}{\sum_{j=1}^N w_j}, \frac{(N-2)w_k}{\sum_{j=1}^N w_j}\right)$$

of the initial problem. ■

7. The Deadlock Problem

7.1. THE PROBLEM

The deadlock problem is a frozen situation generally resulting from a circular inter-process dependence. It often occurs in a wait-for-communication scheme or for a concurrent access to a resource [6].

The problem considers a network of N finite homogeneous Markov chains. With the same state space $\forall k \in \{1, \ldots, N\}$, ${}^k\mathscr{X} = \mathscr{X} = \{1, \ldots, 4\}$, where state 1 = "active", state 2 = "idle", state 3 = "terminated", state 4 = "blocked", and with the following transition matrix:

$$M_{\rho_k} = \begin{pmatrix} \alpha_k & 1-\alpha_k & 0 & 0 \\ \beta_k & \gamma_k & \delta_k & 1-\beta_k-\gamma_k-\delta_k \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

where $\rho_k = (\alpha_k, \beta_k, \gamma_k, \delta_k) \in]0, 1[^4$, and $\beta_k + \gamma_k + \delta_k < 1$.

The Markov chains verify condition (C): each of the chain has only one ergodic class (reduced to the state 4) and has only one transient class. The connection in the network is expressed here by the relations:

$$L_1 \equiv \sum_{k=1}^N \alpha_k - 1 = 0 \tag{3}$$

(from state 1, there is almost surely at least one process which stays in state 1);

$$L_2 \equiv \sum_{k=1}^N \beta_k - 1 = 0 \tag{4}$$

(from state 2, there is almost surely at least one process which goest to state 1);

$$L_3 \equiv \sum_{k=1}^N \gamma_k - 1 = 0 \tag{5}$$

(from state 2, there is almost surely at least one process which stays in that state);

$$L_4 \equiv \sum_{k=1}^{N} \delta_k - 1 = 0 \tag{6}$$

(from state 2, there is almost surely at least one process which goes to state 3). The set of the ρ_k's, $\{\rho_k\}$ such that $\rho_k = (\alpha_k, \beta_k, \gamma_k, \delta_k)$, verifying these constraints is denoted by $\mathcal{R}$.

7.2. DEFINITIONS

The set $\mathcal{S}$ of system states is $\mathcal{X}^N$.
The set $\mathcal{S}^0$ of system initial states is the set $\mathcal{X}^N$ itself.
The set $\mathcal{T}$ of functioning rules is $(M_\rho)_{\rho \in \mathcal{R}}$.
The decision function is:

$$f \equiv \sum_{k=1}^{N} q_k \frac{(\beta_k + \delta_k) + (1 - \alpha_k)(1 + \delta_k)}{(1 - \alpha_k)(1 - \beta_k - \gamma_k - \delta_k)},$$

where the constants q_k are such that $\forall k \in \{1, \ldots, N\}$, $q_k \in]0, 1[$ and $\Sigma_{k=1}^{N} q_k = 1$.

The reason to the choice of this function is that states 1, 2, 3 are transient, while state 4 is an absorbing state; a good criterion is to avoid entering too quickly into state 4. As one can only enter in state 4 from state 2, it is equivalent to say that a *bad* criterion is to enter state 2 as quickly as possible, i.e., to minimize the sum of the respective mean sojourn times in state 1, 2, 3 starting from state 2. As indicated in subsection 3.2, computing $(I - {}^kW)^{-1}$ yields

$$\sum_{i=1}^{3} {}^kS_{2i} = \frac{\beta_k + \delta_k + (1 - \alpha_k)(1 + \delta_k)}{(1 - \alpha_k)(1 - \beta_k - \gamma_k - \delta_k)},$$

from which we get the expression of f.

The search for optimal solutions of f, when taking into account the constraints L_i and introducing Lagrange multipliers λ_i, $i \in \{1, 2, 3, 4\}$ leads to the following system of equations:

$$\frac{\partial f}{\partial \alpha_k} + \frac{\partial}{\partial \alpha_k} \sum_{i=1}^{4} \lambda_i L_i \equiv q_k \frac{\beta_k + \delta_k}{(1 - \alpha_k)^2 (1 - \beta_k - \gamma_k - \delta_k)} + \lambda_1 = 0, \tag{7}$$

$$\frac{\partial f}{\partial \beta_k} + \frac{\partial}{\partial \beta_k} \sum_{i=1}^{4} \lambda_i L_i \equiv \frac{q_k}{1 - \alpha_k} \left[\frac{1 - \gamma_k + (1 - \alpha_k)(1 + \delta_k)}{(1 - \beta_k - \gamma_k - \delta_k)^2} \right] + \lambda_2 = 0, \tag{8}$$

$$\frac{\partial f}{\partial \gamma_k} + \frac{\partial}{\partial \gamma_k} \sum_{i=1}^{4} \lambda_i L_i \equiv \frac{q_k}{1 - \alpha_k} \left[\frac{\beta_k + \delta_k + (1 - \alpha_k)(1 + \delta_k)}{(1 - \beta_k - \gamma_k - \delta_k)^2} \right] + \lambda_3 = 0, \tag{9}$$

$$\frac{\partial f}{\partial \delta_k} + \frac{\partial}{\partial \delta_k} \sum_{i=1}^{4} \lambda_i L_i \equiv \frac{q_k}{1 - \alpha_k} \left[\frac{3 - \beta_k - 2\alpha_k - 2\gamma_k + \alpha_k \beta_k + \alpha_k \gamma_k}{(1 - \beta_k - \gamma_k - \delta_k)^2} \right] + \lambda_4 = 0. \tag{10}$$

The simultaneous solution of all these equations is difficult. So we derive a partial solution, first in relying on the importance of the role of fairness, and second in fixing 2 parameters out of 4: then we have $C_4^2 = 6$ cases to study. Also remark that this study has only a sense for $N \geq 4$; this is due to the condition $\beta_k + \gamma_k + \delta_k < 1$.

THEOREM 7.1. *Suppose that* $\forall k \in \{1, \ldots, N\}$, $\alpha_k = \frac{1}{N}$, $\beta_k = \frac{1}{N}$. *Then the set* $\bar{\mathscr{A}}$ *of bad functioning rules is the set of* M_ρ*'s,* $M_\rho = (M_{\rho_1}, \ldots, M_{\rho_N})$ *where*

$$\rho_k = \left(\frac{1}{N}, \frac{1}{N}, \gamma_k, -\frac{N^2+2N-1}{3N^2-5N+2}\gamma_k + \frac{4N^2-3N+1}{N(3N^2-5N+2)}\right),$$

$k \in \{1, \ldots, N\}$, *when*

$$(\gamma_1, \ldots, \gamma_N) \in \left(\left]0, \frac{4N^2-3N+1}{N(N^2+2N-1)}\right[\right)^N.$$

The fairness belongs to this set. Its complementary set $\mathscr{A}$ *is the set of advisable functioning rules.*

Proof. (a) Considering the equations (9) and (10), we have:

$$\frac{q_k}{\frac{N-1}{N}}\left[\frac{1+\left(2-\frac{1}{N}\right)\delta_k}{\left(\frac{N-1}{N}-\gamma_k-\delta_k\right)^2}\right] + \lambda_3 = 0,$$

$$\frac{q_k}{\frac{N-1}{N}}\left[\frac{3\left(1-\frac{1}{N}\right)+\frac{1}{N^2}+\left(\frac{1-2N}{N}\right)\gamma_k}{\left(\frac{N-1}{N}-\gamma_k-\delta_k\right)^2}\right] + \lambda_4 = 0.$$

Thus

$$\frac{\lambda_3}{\lambda_4} = \frac{1+\left(2-\frac{1}{N}\right)\delta_k}{3-\frac{3}{N}+\frac{1}{N^2}+\left(\frac{1-2N}{N}\right)\gamma_k} = \frac{N^2+2N-1}{3N^2-5N+2},$$

which shows that

$$\delta_k = -\frac{N^2+2N-1}{3N^2-5N+2}\gamma_k + \frac{4N^2-3N+1}{N(3N^2-5N+2)}.$$

Let us denote by $(\hat{\gamma}_k, \hat{\delta}_k)$ these solutions. It can be easily checked that $(\frac{1}{N}, \frac{1}{N})$ is one of such solutions. Note that the q_k's do not appear in these solutions.

(2) Since $\delta_k \in \,]0, 1[$, we have

$$0 < -\frac{N^2+2N-1}{3N^2-5N+2}\gamma_k + \frac{4N^2-3N+1}{N(3N^2-5N+2)} < 1,$$

which implies that

$$\frac{-3N^3 + 9N^2 - 5N + 1}{N(N^2 + 2N - 1)} < \lambda_k < \frac{4N^2 - 3N + 1}{N(N^2 + 2N - 1)} .$$

Since $N \geqslant 4$,

$$\frac{-3N^3 + 9N^2 - 5N + 1}{N(N^2 + 2N - 1)} < 0 ,$$

and

$$\frac{4N^2 - 3N + 1}{N(N^2 + 2N - 1)} < 1 .$$

Therefore the result is valid for

$$(\gamma_1, \ldots, \gamma_N) \in \left(\left] 0, \frac{4N^2 - 3N + 1}{N(N^2 + 2N - 1)} \right[\right)^N .$$

(3) It remains to show that these solutions $(\hat{\gamma}_k, \hat{\delta}_k)$ minimize the function f,

$$f = \frac{N}{N-1} \sum_{k=1}^{N} q_k f_k ,$$

where

$$f_k = \frac{(2N-1)\delta_k + N}{(N-1) - N(\gamma_k + \delta_k)} ,$$

under the constraints (5) and (6) and

$$\gamma_k + \delta_k < \frac{N-1}{N} ,$$

$$\gamma_k > 0, \delta_k > 0, k \in \{1, \ldots, N\} .$$

By setting $u_k = \gamma_k + \delta_k$ and $\rho_k^2 = (2N-1)\delta_k + N$, the problem changes to

$$g = \frac{N}{N-1} \sum_{k=1}^{N} q_k g_k ,$$

where

$$g_k = \frac{\rho_k^2}{(N-1) - Nu_k} ,$$

under the constraints

$$\sum_{k=1}^{N} u_k = 2 ,$$

$$u_k < \frac{N-1}{N} ,$$

$$u_k > 0, \rho_k > 0 , \quad k \in \{1, \ldots, N\} ,$$

$$\sum_{k=1}^{N} \rho_k^2 = N^2 + 2N - 1 .$$

Note that since $N \geqslant 4$, $N^2+2N-1>0$ this polynomial being positive outside $[-1-\sqrt{2}, -1+\sqrt{2}]$. g_k is convex, since its Hessian

$$\nabla^2 g_k = \begin{pmatrix} \dfrac{2}{(N-1)-Nu_k} & \dfrac{2N\rho_k}{[(N-1)-Nu_k]^2} \\ \dfrac{2N\rho_k}{[(N-1)-Nu_k]^2} & \dfrac{2N^2\rho_k^2}{[(N-1)-Nu_k]^3} \end{pmatrix}$$

corresponds to a positive semi-definite quadratic form. Therefore,

$$g = \frac{N}{N-1} \sum_{k=1}^{N} q_k g_k$$

is also convex. Thus, the solutions $(\hat{\gamma}_k, \hat{\delta}_k)$ of the initial problem (which correspond to the optimal solutions $(\hat{u}_k, \hat{\delta}_k)$ of the reduced problem) are solutions which minimize the function f.

THEOREM 7.2. *Suppose that* $\forall k \in \{1, \ldots, N\}$, $\alpha_k = \frac{1}{N}$, $\gamma_k = \frac{1}{N}$. *Then the set* $\bar{\mathscr{A}}$ *of bad functioning rules is the set of* M_ρ*'s*, $M_\rho = (M_{\rho_1}, \ldots, M_{\rho_N})$, *where*

$$\rho_k = \left(\frac{1}{N}, \beta_k, \frac{1}{N}, -\frac{2N+1}{3N-2}\beta_k + \frac{5N-1}{N(3N-2)}\right),$$

$k \in \{1, \ldots, N\}$, *when*

$$(\beta_1, \ldots, \beta_N) \in \left(\left]0, \frac{5N-1}{N(2N+1)}\right[\right)^N.$$

The fairness belongs to this set. Its complementary set $\mathscr{A}$ *is the set of advisable functioning rules.*

Proof. (1) Considering equations (8) and (10), we have:

$$\frac{q_k(2+\delta_k)}{\left(\frac{N-1}{N} - \beta_k - \delta_k\right)^2} + \lambda_2 = 0,$$

$$\frac{q_k}{\frac{N-1}{N}}\left[\frac{-\left(\frac{N-1}{N}\right)\beta_k + 3N - \frac{4}{N} + \frac{1}{N^2}}{\left(\frac{N-1}{N} - \beta_k - \delta_k\right)^2}\right] + \lambda_4 = 0.$$

This gives

$$\frac{\lambda_2}{\lambda_4} = \frac{\left(\frac{N-1}{N}\right)(2+\delta_k)}{-\left(\frac{N-1}{N}\right)\beta_k + 3N - \frac{4}{N} + \frac{1}{N^2}} = \frac{2N+1}{3N-2},$$

and thus,

$$\delta_k = -\frac{2N+1}{3N-2}\,\beta_k + \frac{5N-1}{N(3N-2)}\,.$$

Let us denote by $(\hat{\beta}_k, \hat{\delta}_k)$ these solutions. We can easily check that $(\frac{1}{N}, \frac{1}{N})$ is one of such solutions. As in the case of Theorem 7.1, the coefficients q_k do not appear in the solutions.

(2) Since $\delta_k \in]0, 1[$, we must have

$$0 < -\frac{2N+1}{3N-2}\,\beta_k + \frac{5N-1}{N(3N-2)} < 1\,,$$

which implies

$$\frac{-3N^2+7N-1}{N(2N+1)} < \beta_k < \frac{5N-1}{N(2N+1)}\,.$$

Since $N \geq 4$, $-3N^2+7N-1<0$ (this polynomial being negative for $N \not\in [\frac{7-\sqrt{37}}{6}, \frac{7+\sqrt{37}}{6}]$ and $\frac{5N-1}{N(2N+1)} < 1$). Thus, the proposition is valid for $(\beta_1, \ldots, \beta_N) \in (]0, \frac{5N-1}{N(2N+1)}[)^N$.

(3) It remains to show that these solutions $(\hat{\beta}_k, \hat{\delta}_k)$ minimize the function f which is here

$$f = \frac{N}{N-1} \sum_{k=1}^{N} q_k f_k\,,$$

where

$$f_k = \frac{N\beta_k + (2N-1)\delta_k + (N-1)}{(N-1) - N(\beta_k + \delta_k)}\,,$$

under constraints (3) and (5), and

$$\beta_k + \delta_k < \frac{N-1}{N}\,,$$

$$\beta_k > 0\,, \quad \delta_k > 0\,, \quad k \in \{1, \ldots, N\}\,.$$

By writing f_k in the form

$$f_k = -1 + \frac{(N-1)\delta_k + 2(N-1)}{(N-1) - N(\beta_k + \delta_k)}$$

and changing variables (as in (7.1)) $\mu_k = \beta_k + \delta_k$ and $\rho_k^2 = (N-1)\delta_k + 2(N-1)$, we transform the initial problem into the minimization problem of

$$g = \frac{N}{N-1} \sum_{k=1}^{N} q_k g_k\,,$$

where

$$g_k = \frac{\rho_k^2}{(N-1) - Nu_k}\,,$$

under the constraints

$$\sum_{k=1}^{N} u_k = 2 ,$$

$$u_k < \frac{N-1}{N} ,$$

$$u_k > 0 ,$$

$$\rho_k > 0 , \quad k \in \{1, \ldots , N\} ,$$

$$\sum_{k=1}^{N} \rho_k^2 = (2N+1)(N-1) .$$

The function g_k is convex since its Hessian $\nabla^2 g_k$ is the same as the one of Theorem 7.1. Thus g is convex. The solutions $(\hat{\beta}_k, \hat{\delta}_k)$ of the initial problem correspond to the optimal solutions $(\hat{u}_k, \hat{\rho}_k)$ of the transformed problem which minimize f. In the particular case where $\forall k \in \{1, \ldots , N\}$, $q_k = \frac{1}{N}$, we are brought back to the study completed in [2]. ∎

THEOREM 7.3. *Suppose that* $\forall k \in \{1, \ldots , N\}$, $\alpha_k = \frac{1}{N}$, $\delta_k = \frac{1}{N}$. *Then the set* $\bar{\mathscr{A}}$ *of bad functioning rules is the set of* M_ρ*'s,* $M_\rho = (M_{\rho_1}, \ldots , M_{\rho_N})$, *where*

$$\rho_k = \left(\frac{1}{N}, \beta_k, \frac{-2N^2+N+1}{N^2+2N-1}\,\beta_k + \frac{3N^2+N-2}{N(N^2+2N-1)}, \frac{1}{N}\right),$$

$k \in \{1, \ldots , N\}$, *when*

$$(\beta_1, \ldots , \beta_N) \in \left(\left]0, \frac{3N^2+N-2}{N(N^2-N-1)}\right[\right)^N .$$

The fairness belongs to this set. Its complementary set $\mathscr{A}$ *is the set of advisable functioning rules.*

Proof. (1) Considering the equations (8) and (9), we have

$$\frac{q_k}{\frac{N-1}{N}}\left[\frac{-\gamma_k + \frac{2N^2-1}{N^2}}{\left(\frac{N-1}{N} - \beta_k - \gamma_k\right)^2}\right] + \lambda_2 = 0 ,$$

$$\frac{q_k}{\frac{N-1}{N}}\left[\frac{-\beta_N + \frac{N^2+N-1}{N^2}}{\left(\frac{N-1}{N} - \beta_k - \gamma_k\right)^2}\right] + \lambda_3 = 0 ,$$

which gives

$$\frac{\lambda_2}{\lambda_3} = \frac{\gamma_k - \frac{2N^2-1}{N^2}}{\beta_k + \frac{N^2+N-1}{N^2}} = \frac{-(N-1)(2N+1)}{N^2+2N-1} .$$

Thus

$$\gamma_k = \frac{-2N^2+N+1}{N^2+2N-1}\,\beta_k + \frac{3N^2+N-2}{N(N^2+2N-1)}\,.$$

Let us denote by $(\hat{\beta}_k, \hat{\gamma}_k)$ these solutions. It can be easily checked that $(\frac{1}{N}, \frac{1}{N})$ is one of these solutions. Note here again that the q_k do not appear in the solutions.

(2) Since $\gamma_k \in]0, 1[$, we must have

$$0 < \frac{-2N^2+N+1}{N^2+2N-1}\,\beta_k + \frac{3N^2+N-2}{N(2N^2-N-1)} < 1$$

This implies that

$$\frac{-N^2+2}{N(2N+1)} < \beta_k < \frac{3N^2+N-2}{N(2N^2-N-1)}\,.$$

Since $N \geq 4$ and

$$\frac{3N^2+N-2}{N(2N^2-N-1)} < 1,\ \frac{-N^2+2}{N(2N+1)} < 0\,.$$

Thus, the proposition is valid for

$$(\beta_1, \ldots, \beta_N) \in \left(\left]0, \frac{3N^2+N-2}{N(2N^2-N-1)}\right[\right)^N\,.$$

(3) Let us show that the solutions $(\hat{\beta}_k, \hat{\gamma}_k)$ minimize the function f which is here:

$$f = \frac{1}{N-1}\sum_{k=1}^{N} q_k f_k\,,$$

where

$$f_k = \frac{N^2\beta_k + (N^2+N-1)}{(N-1) - N(\beta_k + \gamma_k)}\,,$$

under the constraints (4) and (5), and

$$\beta_k + \gamma_k < \frac{N-1}{N}\,,$$

$$\beta_k > 0\,,\quad \gamma_k > 0\,,\quad k \in \{1, \ldots, N\}\,.$$

First remark that since $N \geq 4$, $N^2 + N - 1 > 0$ (because this polynomial is positive outside the interval $[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}]$). By setting $u_k = \beta_k + \gamma_k$ and $\rho_k^2 = N^2\beta_k + (N^2 + N - 1)$, the problem consists then in minimizing

$$g = \frac{1}{N-1}\sum_{k=1}^{N} q_k g_k\,,$$

where

$$g_k = \frac{\rho_k^2}{(N-1) - Nu_k},$$

under the constraints

$$\sum_{k=1}^{N} u_k = 2,$$

$$u_k < \frac{N-1}{N},$$

$$u_k > 0, \quad \rho_k > 0, \quad k \in \{1, \ldots, N\},$$

$$\sum_{k=1}^{N} \rho_k^2 = N(N^2 + 2N - 1) > 0.$$

Here again, we see that the function g_k is convex since its Hessian is the same as in the preceding cases. Thus g is convex. Therefore, the solutions $(\hat{\beta}_k, \hat{\gamma}_k)$ of the initial problem (which correspond to optimal solutions $(\hat{u}_k, \hat{\rho}_k)$ of the reduced problem) are solutions which minimize the function f. ■

THEOREM 7.4. *Suppose that* $\forall k \in \{1, \ldots, N\}$, $\beta_k = \frac{1}{N}$, $\delta_k = \frac{1}{N}$ *and*

$$q_k < \frac{N-2}{N(N-3)}.$$

Then the set $\mathscr{A}$ *of bad functioning rules is reduced to* $\{M_\rho\} = \{(M_{\rho_1}, \ldots, M_{\rho_N})\}$, *where*

$$\rho_k = \left(\frac{1}{N}, \frac{1}{N}, -(N-3)q_k + \frac{N-2}{N}, \frac{1}{N}\right),$$

$k \in \{1, \ldots, N\}$. *In the particular case where* $\forall k \in \{1, \ldots, N\}$, $q_k = \frac{1}{N}$, *we find the fairness solution* $\rho_k = (\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N})$. $\mathscr{A}$ *is the set of advisable functioning rules.*

Proof. (1) Considering the equations (7) and (9), we have:

$$\frac{2q_k}{N(1-\alpha_k)^2\left(\frac{N-2}{N} - \gamma_k\right)} + \lambda_1 = 0,$$

$$\frac{-q_k[1 + (1-\alpha_k)(N+1)]}{N(1-\alpha_k)\left(\frac{N-2}{N} - \gamma_k\right)} + \lambda_2 = 0,$$

which gives

$$\frac{\lambda_1}{\lambda_2} = \frac{-2}{(1-\alpha_k)[2 + (1-\alpha_k)(N+1)]} = \frac{-2N}{2(N-1) + (N+1)\Sigma_{j=1}^{N}(1-\alpha_j)^2}.$$

That is to say $\forall j, k \in \{1, \ldots, N\}$,

$$\frac{-2}{(N+3) - 2(N+2)\alpha_k + (N+1)\alpha_k^2} = \frac{-2}{(N+3) - 2(N+2)\alpha_j + (N+1)\alpha_j^2}.$$

This yields that

$$(\alpha_k - \alpha_j)[-2(N+2) + (N+1)(\alpha_k + \alpha_j)] = 0 .$$

Since α_k, $\alpha_j \in]0, 1[$,

$$-2(N+2) + (N+1)(\alpha_k + \alpha_j) < -2(N+2) + 2(N+1) = -2 < 0 ,$$

then

$$\alpha_k = \alpha_j = \frac{1}{N} .$$

Substituting $\alpha_k = \frac{1}{N}$ in (7), we also have

$$-\lambda_1 = \frac{2q_k}{N\left(\frac{N-1}{N}\right)^2\left(\frac{N-2}{N} - \gamma_k\right)} = \frac{2N}{(N-1)^2(N-3)} ,$$

which gives

$$\gamma_k = -(N-3)q_k + \frac{N-2}{N} .$$

Let us denote this solution by $(\hat{\alpha}_k, \hat{\gamma}_k)$.

(2) Since $\gamma_k \in]0, 1[$, we must have

$$0 < -(N-3)q_k + \frac{N-2}{N} < 1 ,$$

which yields

$$\frac{-2}{N(N-3)} < q_k < \frac{N-2}{N(N-3)} .$$

Since

$$N \geqslant 4, \frac{-2}{N(N-3)} < 0$$

and

$$0 < \frac{N-2}{N(N-3)} < 1 .$$

Thus, we have the condition

$$q_k < \frac{N-2}{N(N-3)} .$$

(3) Let us show now that the solution $(\hat{\alpha}_k, \hat{\gamma}_k)$, $k \in \{1, \ldots, N\}$, minimizes f, which is here

$$f = \sum_{k=1}^{N} q_k f_k ,$$

where

$$f_k = \frac{\frac{2}{N} + (1-\alpha_k)\frac{N+1}{N}}{(1-\alpha_k)\left(\frac{N-2}{N} - \gamma_k\right)},$$

under constraints (3) and (5), and

$$\alpha_k \in]0, 1[,$$
$$0 < \gamma_k < \frac{N-2}{N}, \quad k \in \{1, \dots, N\}.$$

If we set

$$a = \frac{2}{N}, b = \frac{N+1}{N}, c = \frac{N-2}{N},$$

the Hessian of f_k is

$$\nabla^2 f_k = \begin{pmatrix} \frac{2a}{(1-\alpha_k)^3(c-\gamma_k)} & \frac{a}{(1-\alpha_k)^2(c-\gamma_k)^2} \\ \frac{a}{(1-\alpha_k)^2(c-\gamma_k)^2} & \frac{2[a+b(1-\alpha_k)]}{(1-\alpha_k)(c-\gamma_k)^3} \end{pmatrix}.$$

It corresponds to a positive semi-definite quadratic form, since $\forall (x_1, x_2) \in \boldsymbol{R}^2$

$$(x_1\ x_2)\nabla^2 f_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2a}{(1-\alpha_k)^3(c-\gamma_k)}\left[\left(x_1 + \frac{1}{2\frac{c-\gamma_k}{1-\alpha_k}} x_2\right)^2 + \frac{[\frac{3}{4}a + b(1-\alpha_k)]}{a\left(\frac{c-\gamma_k}{1-\alpha_k}\right)^2} x_2^2\right] \geqslant 0$$

This shows that f_k is convex and consequently f is convex.
The solution $(\hat{\alpha}_k, \hat{\gamma}_k)$, $k \in \{1, \dots, N\}$ corresponds to a minimum for f. ■

7.3. OTHER CASES

REMARK 7.1. (*concerning the case where* $\forall k \in \{1, \dots, N\}$, $\beta_k = \frac{1}{N}$, $\gamma_k = \frac{1}{N}$). *Here the set* $\bar{\mathscr{A}}$ *of bad functioning rules is a subset of the set of* M_ρ*'s*, $M_\rho = (M_{\rho_1}, \dots, M_{\rho_N})$, *where*

$$\rho_k = \left(\alpha_k, \frac{1}{N}, \frac{1}{N}, \delta_k\right),$$

$k \in \{1, \dots, N\}$, *are such that*

$$(\alpha_k, \delta_k) \in]0, 1[\times \left]0, \frac{N-2}{N}\right[$$

for which the following relation holds:

$$\frac{(N-2)+N(N-3)\delta_k - N^2\delta_k^2}{N(N-1)(3-5\alpha_k+2\alpha_k^2)} = \frac{N(2N-5)-N^2\,\Sigma_{k=1}^N\,\delta_j^2}{3N^2(N-1)-5N(N-1)+2N(N-1)\,\Sigma_{j=1}^N\,\alpha_j^2}\,.$$

The fairness $(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N})$ *belongs to this set of* M_ρ*'s.*

Sketch of Proof. Considering the equations (7) and (10), we have:

$$q_k\,\frac{\frac{1}{N}+\delta_k}{(1-\alpha_k)^2\left(\frac{N-2}{N}-\delta_k\right)}+\lambda_1=0\,,$$

and

$$q_k\,\frac{\frac{2N-3}{N}+2\frac{N-1}{N}\alpha_k}{(1-\alpha_k)\left(\frac{N-2}{N}-\delta_k\right)^2}+\lambda_4=0\,,$$

thus

$$\frac{\lambda_1}{\lambda_4}=\frac{\left(\frac{1}{N}+\delta_k\right)\left(\frac{N-2}{N}-\delta_k\right)}{(1-\alpha_k)\left(\frac{3N-3}{N}+\frac{2(N-1)}{N}\delta_k\right)}\,.$$

This leads to the above relation. ∎

REMARK 7.2. (*concerning the case where* $\forall k\in\{1,\ldots,N\}$, $\gamma_k=\frac{1}{N}$, $\delta_k=\frac{1}{N}$). *As in the previous remark, the set* $\mathscr{A}$ *of bad functioning rules is a subset of the set of* M_ρ's, $M_\rho=(M_{\rho_1},\ldots,M_{\rho_N})$, *where*

$$\rho_k=\left(\alpha_k,\,\beta_k,\,\frac{1}{N},\,\frac{1}{N}\right)$$

$k\in\{1,\ldots N\}$, *are such that*

$$(\alpha_k,\,\beta_k)\in\,]0,\,1[\,\times\left]0,\,\frac{N-2}{N}\right[$$

for which the following relation holds:

$$\frac{(N-2)+N(N-3)\beta_k-N^2\beta_k^2}{2N-(3N+1)\alpha_k+(N+1)\alpha_k^2}=\frac{2N-5-N\,\Sigma_{j=1}^N\,\beta_j^2}{2N^2-3N-1+(N+1)\,\Sigma_{j=1}^N\alpha_j^2}\,.$$

The fairness $(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N})$ *belongs to this set of* M_ρ*'s.*

Sketch of Proof. Considering the equations (7) and (8), we have:

$$q_k \frac{\beta_k + \frac{1}{N}}{(1-\alpha_k)^2\left(\frac{N-2}{N} - \beta_k\right)} + \lambda_1 = 0$$

and

$$q_k \frac{\frac{N-1}{N} + \frac{N+1}{N}(1-\alpha_k)}{(1-\alpha_k)\left(\frac{N-2}{N} - \beta_k\right)^2} + \lambda_2 = 0\,,$$

then

$$\frac{\lambda_1}{\lambda_2} = \frac{\left(\beta_k + \frac{1}{N}\right)\left(\frac{N-2}{N} - \beta_k\right)}{(1-\alpha_k)\left[\frac{N-1}{N} + \frac{N+1}{N}(1-\alpha_k)\right]}\,,$$

this leads to the above relation. ∎

8. Conclusion

We have introduced a model for distributed computing that helps to handle the complexity of concurrency control problems.

With this approach, we first showed that one can formally find good functioning properties (safety and liveness properties) in a systematic way for the considered distributed control problems. It is an efficient theoretical tool for reasoning about distributed algorithms, with property that the global behaviour is obtained from the study of local behaviour.

We then devised optimal functioning properties for different classical distributed computing problems in which fairness is proven a decisive issue.

On the other hand, as we are able to estimate statistically the model's parameters from various executions ([3]), this gives tools for the evaluation of the means for the evaluation of the performance of distributed algorithms in their average behaviour. This also corresponds to the requirements for a self-tuning method for distributed systems. From a practical point of view, we have encoded the model's functionalities in a simulator, having thus an interesting practical tool for comparing different asynchronous distributed algorithms ([3]).

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